Remarks on 3 -prime near-ring involving * - involution

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Abstract

This work introduces the concept of * - involution in 3 - prime near-ring N together with its semi group ideal S and it establishes some results on N as well as S involving * - involution. In addition, examples are given to demonstrate the essentialities of 3- primeness in the hypothesis of our theorems. Finally, we conclude it with some open problems.

Keywords: Ring, near-ring, 3 -prime near-ring, * - involution, semi group ideal

1. Introduction

By a right near-ring we shall mean a non-empty set N endowed with two associative operations called addition (+) and multiplication (denoted by (⋅) and (∗) respectively) satisfying the following conditions

(i) (N, +) is an additive group (not necessarily abelian)
(ii) (N, ∗) is a semi group
(iii) Multiplication(⋅) distributes over addition(+) from the right (denoted by

\[(x + y) ∗ z = xz + yz \forall x, y, z ∈ N\]

A right near-ring N is said to be zero symmetric if x ∗ 0 = 0 ∅ x ∈ N(evolving that right distributive gives 0 ∗ x = 0). Eviduing that N is said to be 3 prime near-ring, will have the property that aNb = {0} for a, b ∈ N implies a = 0 or b = 0. Normal subgroup S of (N, +) is said to be an ideal of N if SN ⊆ S and a(b + s) − ab ∈ S for s ∈ S and a, b ∈ N.

A map ∗: N → N is said to be *-involution if for x, y ∈ N,

(i) (x + y)∗ = x∗ + y∗, (ii) (xy)∗ = x∗y∗, (iii) (x∗)∗ = x.

A near-ring N equipped with an *-involution is called a near-ring with *-involution or *- near-ring. We refer the reader to the books of Clay [6], Meldrum [9] and Pilz [11] for the near-ring theory and its applications. Recall that a near-ring N is called 0 - prime if the product of any two of its ideals is non-zero. In addition, a near-ring N is called 3 - prime if for any non-zero x, y ∈ N, xNy ≠ {0} [7, 12]. Posner published his paper [13] in 1957; various authors have investigated the properties of derivations of prime and semi prime rings. Existence important ring theory tools [4], these outcomes are one of the sources of the developments of such theories as the theory of differential identities [8] and the theory of Hopf algebra action on rings [8], [10]. The study of derivations of near-rings was initiated by Bell and Mason in 1997 [2], but up to now only a few papers on 3-prime near-rings were published.

Bell, Boua, and Oukhtite [4] generalized some results known in this field involving the semi group ideal instead of entire near-rings. From these observations, one can ask a natural question “Can one apply the *-involution on the structure of a 3 – prime near-ring N and its semi group ideal S? The aim of this paper is to give an affirmative answer to this question. In Section 2, we establish that a 3- prime near-ring N with * -involution is an associative ring (or simply a ring). Section 3, devotes the result on semi group ideal of N with *-involution becomes a ring. Also, we construct an example which establishes that our Theorems do not hold even for simple 0-prime near-rings with a right identity element.

2. On 3- prime near-ring with * - involution

In this section, we establish the following result.

Theorem 2.1

Let N be a 3- prime near-ring with * -involution. Then N
is a ring.

**Proof**

Assume that * is an involution (−involution) on \( N \). We claim that \( N \) is a ring. We break the proof in two steps.

**Step 1**

We prove the multiplication on \( N \) satisfies left distributive law, that is

\[
x(y + z) = xy + xz \quad \text{for all } x, y, z \in N \tag{2.1}
\]

Using the properties (iii), (ii) and (i) in the definition of −involution and right distributive law, we have

\[
x(y + z) = ((x(y + z))^*)^* = ((y + z)^*x^*) = ((y^* + z^*)x^*) = (y^*x^* + z^*x^*) = ((xy)^*)^* + ((xz)^*)^* = xy + xz.
\]

This completes the proof of Step 1.

**Step 2**

We show that addition on \( N \) is abelian (viz: \((N, +)\) is abelian)

Replace \( x \) by \((y + z)\) and y and z by w in the relation (2.1) to get

\[
(y + z)(w + w) = (y + z)w + (y + z)w
\]

for any \( w, y, z \in N \).

\[
(y + z)(w + w) = yw + zw + yw + zw
\]

for any \( w, y, z \in N \). \( \tag{2.2} \)

\[
(y + z)(w + w) = y(w + w) + z(w + w)
\]

for any \( w, y, z \in N \).

\[
(y + z)(w + w) = yw + yw + zw + zw
\]

for any \( w, y, z \in N \). \( \tag{2.3} \)

Combining the relations (2.2) and (2.3), we find that

\[
yw + zw + yw + zw = yw + yw + zw + zw
\]

for any \( w, y, z \in N \).

\[
zw + yw = yw + zw
\]

for any \( w, y, z \in N \).

\[
(z + y) - (y + z)w = 0
\]

for all \( w, y, z \in N \).

This implies that \((z + y) - (y + z))N = \{0\}\) for all \( y, z \in N \). \( \tag{2.4} \)

In view of the result of Bell and Mason [2, Lemma 1.2 (i)], and relation (2.4), we have

\[
(z + y) - (y + z) = 0.
\]

Hence \((N, +)\) is an additive abelian group. From Step 1 and Step 2, we see that a 3-prime near-ring \( N \) becomes a ring.

**Remark 2.3**

The following example shows that the condition of 3-prime near-ring in Theorem 2.1 is essential.

**Example 2.4**

Take a non-commutative near-ring \( M \) and define

\[
N = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{N} \right\}, \text{ and a map } *: N \rightarrow N
\]

by \( \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) for all \( x, y \in M \).

Then * is an involution (−involution) on \( N \), but \( N \) is neither a 3-prime near-ring nor a ring. For instance

For \( *-\text{involution on } N \)

Condition (i) \((x + y)^* = x^* + y^* \) and (ii) \((x^*)^* = x \),

where \( x = \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) and \( y = \begin{pmatrix} 0 & y_1 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), for all \( x_1, x_2, y_1, y_2 \in S \), are straightforward.

(ii) \((xy)^* = \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & y_1 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

Implies \((xy)^* = y^* x^* \).

**N is not a 3-prime near-ring**

We have

\[
xNy = \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & y_1 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{but } x \neq 0 \text{ and } y \neq 0.
\]

From the above observations, one can easily see that \( N \) is not a ring.
3. Semi group ideal with * – involution

We begin with the following definition

Definition 3.1

A non-empty subset S of N is called semi group right ideal (resp. semi group left ideal) of N if SN ⊆ N (resp. NS ⊆ N); and S is said to be a semi group ideal if it is both a right semi group ideal as well as a left semi group ideal of N.

Example 3.2

Let \( N = \{0, \alpha, \beta, \gamma\} \) with addition and multiplication tables defined as follows.

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( \alpha )</td>
<td>( \beta )</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>( \alpha )</td>
<td>0</td>
<td>( \gamma )</td>
<td>( \beta )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>( \beta )</td>
<td>( \gamma )</td>
<td>0</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>( \gamma )</td>
<td>( \beta )</td>
<td>( \alpha )</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
\begin{array}{c|cccc}
+ & 0 & \alpha & \beta & \gamma \\
\hline
0 & 0 & \alpha & \beta & \gamma \\
\alpha & \alpha & 0 & \gamma & \beta \\
\beta & \beta & \gamma & 0 & \alpha \\
\gamma & \gamma & \beta & \alpha & 0 \\
\end{array}
\]

\[
\cdot & 0 & \alpha & \beta & \gamma \\
\hline
0 & 0 & 0 & 0 & 0 \\
\alpha & 0 & \alpha & \alpha & \alpha \\
\beta & 0 & \alpha & \beta & \beta \\
\gamma & 0 & \alpha & \gamma & \gamma \\
\end{array}
\]

Theorem 3.1

Let \( N \) be a 3-prime near-ring and \( S \) a semi group ideal. In addition, if \( S \) admits 

\[ * - \text{Involution} \] 

then \( N \) is a ring.

In order to prove this theorem, we first state the result, due to Bell [2].

Fact 3.2

Let \( S \) be a non-zero semi group ideal of a 3-prime near-ring \( N \) with \( x \in N \); given \( xS = \{0\} \) or \( Sx = \{0\} \) then \( x = 0 \).

Proof of Theorem 3.1

Keeping in mind the proof of Step 1 for entire 3-prime near-ring \( N \), for the sake of convenience, we prove it for every \( a, b, c \) in semi group ideal \( S \) of \( N \).

\[ a(b + c) = ((a + b)c)^* = ((b + c)a)^* = (b^*a^* + c^*a^*)^* = (b^*a^*)^* + (c^*a^*)^* = a^*b^* + a^*c^*. \]

This implies that

\[ a(b + c) = ab + ac \quad \text{for all} \quad a, b, c \in S. \quad (3.1) \]

Replacing \( mb \) for \( b \) and \( nb \) for \( c \) in (3.1), we get

\[ a(mb + nb) = amb + anb \quad \text{for all} \quad a, b, c \in S. \]

\[ [a(m + n) - (am + an)]b = 0 \quad \text{for all} \quad a, b, c \in S, \]

where \( m, n \in N \). But, for all \( b \in S \), also

\[
[\alpha(m + n) - (an + am)]S = \{0\} \quad (3.2)
\]

Using Fact 3.2 and (3.2), we find that

\[ l(m + n) = lm + ln \quad \forall l, m, n \in N \]

Hence, the multiplications of \( N \) satisfies left distributive law, \( (N, +) \) is an additive abelian group from Step 2 of Theorem 2.1. □

Corollary 3.3

Let \( N \) be a 3-prime near-ring and \( S \) is a non-zero ideal of \( N \). If \( S \) admits \( * - \text{involutions} \), then \( N \) is a ring.

Proof of the Corollary 3.3 follows immediately from Theorem 3.1. □

Remark 3.4

We construct an example which shows that Theorem 3.1 does not hold even for simple 0-prime near-rings with a right identity element.

Example 3.5

Suppose that \( M \) be a linear space with a basis \( B = \{e_M, e_{e_2}, e_3, \ldots, e_m\} \) over a field \( K \) of characteristic \( \neq 2 \). Define a multiplication \( \cdot : M \times M \to M \) by the rule

\[ m n = 0 \quad \text{for all} \quad m, n \in M \quad \text{with} \quad n \notin \{e_M, -e_M\} \quad \text{and} \quad m e_M = m, \quad m (-e_M) = -m. \]

It is easily seen that \( M \) is a right near-ring. Also \( M \) is a zero symmetric right near-ring with respect to this multiplication (See [1]).

Next, we show that \( M \) is a near-ring with the right identity \( e_M \). Take a non-zero semi group ideal \( S \) of \( M \). Let \( e_M \in S \). Then \( M = Me_M \subseteq S \). This is a contradiction. Thus \( e_M \notin S \). If \( n \in S \), then either \( n + e_M \neq -e_M \) or \( n + (-e_M) \neq e_M \). From the first case, it is easily seen that \( e_M + n \neq e_M \). Thus \( m (e_M + n) = 0 \) for all \( m \in M \), since \( S \) is a semi group ideal, we write \( m = m (e_M + n) - m e_M \in S \), for all \( m \in M \). This implies that \( M \subseteq S \), a contradiction. Hence \( M \) is a right near-ring with identity \( e_M \). Trivially, \( M \) is not a ring.

4. Open questions

In retrospect, we would like to open the questions for further studies as given below.

Question 1: Can the hypothesis that 3-prime be removed from the assumptions in Theorem 2.1 and Theorem 3.1?

Question 2: Can the hypothesis that semi group ideal be removed from the assumptions in Theorem 3.1?

Question 3: Can the hypothesis that \( * - \text{inversion} \) be removed from the assumptions in Theorem 2.1 and Theorem 3.1?

References


